

# Evolution of the Spiral Waves in Excitable System

Ji-Rong Ren,<sup>\*</sup> Tao Zhu,<sup>†</sup> and Shu-Fan Mo<sup>‡</sup>

*Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China*

(Dated: November 3, 2008)

Spiral wave, whose rotation center can be regarded as a point defect, widely exists in various two dimensional excitable systems. In this paper, by making use of *Duan's topological current theory*, we obtain the charge density of spiral waves and the topological inner structure of its topological charge. The evolution of spiral wave is also studied from the topological properties of a two-dimensional vector field. The spiral waves are found generating or annihilating at the limit points and encountering, splitting, or merging at the bifurcation points of the two-dimensional vector field. Some applications of our theory are also discussed.

PACS numbers: 03. 65. Vf, 02. 10. Kn, 82. 40. Ck

## I. INTRODUCTION

Rotating of spiral waves is a fascinating example of self-organization in various two dimensional excitable systems including Belousov-Zhabotinskii reaction[1, 2, 3], cardiac muscle[4], and CO oxidation on platinum surfaces[5]. Recently, much effort has been put into the development of the analytic description of excitable system, and the spiral waves have drawn great interest and have been studied intensively in many ways[6, 7, 8, 9, 10, 11].

The rotation centers of the spiral wave can be regarded as a point defect and defined in terms of a phase singularity. The control of these singular points is a very complex problem in all two-dimensional excitable system. The dynamics of spiral wave are determined not only by properties of the excitable system but also by the topological characteristic number of the spiral wave. The topological charge, which describes the strength of spiral wave phase singularity, is a basic topological number of spiral wave[6, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. It plays an essential role during the processes of creating or annihilating new spiral waves. These processes can be regarded as topological phenomena in excitable system. The mathematical description of these spiral waves relates to the notion of phase which in turn allows one to characterize spiral waves by an index. The phase field  $\theta(r, t)$  of spiral waves is a continuously differentiable function except at the singular points[6, 13, 23, 25]. The sites of spiral wave cores are just the phase singular points. An index, which counts the winding number of phase field around the singular point, is just the topological charge of the spiral wave. From this description, a number of topological arguments have been applied to help us to understand the topological properties of spiral wave and some important topological constraints on behaviors of spiral wave have been investigated[6, 12, 13, 14, 15, 17,

18, 19, 20, 21, 22, 23, 24, 25, 26]. However, the traditional topological theory of spiral wave is not so powerful at directly computing the singularities of phase fields, but only considers a closed path around the studied region, and uses the single-value condition to indirectly analyze the existence of spiral wave in the region. Therefore, it is very important to develop a new method which can directly deduce these spiral waves in a precise topological form, and can reveal the inner structures and the branch processes (generate, annihilate, encounter, split, or merge) of the spiral waves.

In this paper, by making use of *Duan's topological current theory*[27, 28, 29, 30, 31], we study the inner structure of spiral wave in details. It is showed that the center of spiral wave is not only classified by topological charge, but also classified by Hopf index, Brouwer degree in more detail in geometry. We also investigate the evolution of spiral wave. And one sees that the spiral waves generate or annihilate at the limit points and encounter, split, or merge at the bifurcation points of the spiral wave field. Furthermore, we summarize the topological constraints on the behaviors of spiral waves during the above processes. Some possible applications of our topological theory of spiral wave are also discussed.

## II. THE TOPOLOGICAL CHARGE DENSITIES OF SPIRAL WAVES

Excitable system is typically described in terms of reaction-diffusion models. We chose to work with a general two-variable reaction-diffusion system which mathematical description in terms of a nonlinear partial differential equation. This equation is written as

$$\begin{aligned}\partial_t u &= F_1(u, v) + D_u \nabla^2 u, \\ \partial_t v &= F_2(u, v) + D_v \nabla^2 v.\end{aligned}\tag{1}$$

Here  $u$  and  $v$  represent the concentrations of the reagents;  $F_1(u, v)$  and  $F_2(u, v)$  are the reaction functions. The diffusion coefficients of these reagents are  $D_u$  and  $D_v$ . Following the description in Ref.[12], we define a complex function  $Z = \phi^1 + i\phi^2$ , where  $\phi^1 = u - u^*$  and  $\phi^2 = v - v^*$ .

<sup>\*</sup>Email: renjr@lzu.edu.cn.

<sup>†</sup>Corresponding author. Email :zhut05@lzu.cn.

<sup>‡</sup>Email: meshf07@lzu.cn.

Here  $u^*$  and  $v^*$  are the concentrations of the core of spiral wave.

As pointed out in Ref.[12, 15], the sites of the spiral waves are just the isolated zero points of the complex function  $Z = \phi^1 + i\phi^2$ , i.e., the spiral wave field in two-dimensional surface, and in general, it is called the spiral core. The phase field of spiral wave is defined as the argument of the complex function  $Z$ , i.e.,  $\theta(r, t) = \arg(Z) = \arctan(\phi^1/\phi^2)$ . It is easy to see that the zero points of  $Z$  are just the phase singular points of phase field  $\theta(r, t)$ . By using of topological viewpoints, Liu et al.[16] derived a topological expression of the charge density, which is written as

$$\rho(x, y, t) = \sum_{l=1}^N W_l \delta^2[r - r_l(t)], \quad (2)$$

where  $W_l$  is the topological charge of the  $l$ -th spiral wave. This expression also reveals the topological structure of the spiral wave. In our topological theory of spiral wave, the topological structure of spiral wave will play an essential role. In order to make the background of this paper clear, in this section we rewrite the charge density (2) of spiral wave as a topological current.

Now we begin to derive the topological current form of the charge density of the spiral core. We know that the complex function  $Z = \phi^1 + i\phi^2$  can be regarded as the complex representation of a two-dimensional vector field  $\vec{Z} = (\phi^1, \phi^2)$ . Let us define the unit vector:  $n^a = \frac{\phi^a}{\|\phi\|}$  ( $a = 1, 2$ ;  $\|\phi\|^2 = \phi^a \phi^a = Z^* Z$ ). It is easy to see that the zeros of  $Z$  are just the singularities of  $\vec{n}$ . Using this unit vector  $\vec{n}$ , we define a two dimensional topological current

$$J^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b, \quad i, j, k = 0, 1, 2. \quad (3)$$

Applying *Duan's topological current theory*, one can obtain

$$J^i = \delta^2(\vec{\phi}) D^i\left(\frac{\phi}{x}\right), \quad (4)$$

where the Jacobian  $D^i(\frac{\phi}{x})$  is defined as

$$D^i\left(\frac{\phi}{x}\right) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \phi^a \partial_k \phi^b. \quad (5)$$

The delta function expression (4) of the topological current  $J^i$  tell us that it does not vanish only when the spiral waves exist, i.e.,  $Z = 0$ . The sites of the spiral core determine the nonzero solutions of  $J^i$ . The implicit function theorem [32] shows that under the regular condition

$$D^0\left(\frac{\phi}{x}\right) \neq 0, \quad (6)$$

the general solutions of

$$\phi^a(x^0, x^1, x^2) = 0, \quad a = 1, 2, \quad (7)$$

can be expressed as

$$\begin{aligned} \vec{x} &= \vec{z}_l(t), \quad l = 1, 2, \dots, N, \\ x^0 &= t. \end{aligned} \quad (8)$$

From Eq.(7), it is easy to prove that

$$D^i\left(\frac{\phi}{x}\right)\Big|_{z_l} = D\left(\frac{\phi}{x}\right)\Big|_{z_l} \frac{dx^i}{dt}. \quad (9)$$

According to the  $\delta$ -function theory [33] and *Duan's topological current theory*, one can prove that

$$J^i = \sum_{l=1}^N \beta_l \eta_l \delta^2(\vec{x} - \vec{z}_l) \frac{dx^i}{dt}\Big|_{z_l}, \quad (10)$$

in which the positive integer  $\beta_l$  is the Hopf index and  $\eta_l = \text{sgn}(D(\frac{\phi}{x})\Big|_{z_l}) = \pm 1$  is the Brouwer degree. Now the density of spiral wave are expressed in terms of the complex function  $Z$ :

$$\begin{aligned} \rho = J^0 &= \frac{1}{2\pi} \epsilon^{0jk} \epsilon_{ab} \partial_j n^a \partial_k n^b \\ &= \sum_{l=1}^N \beta_l \eta_l \delta^2(\vec{x} - \vec{z}_l). \end{aligned} \quad (11)$$

This is just the charge density of spiral in Ref.[12]. Therefore, the total charge of the system given can be rewritten as

$$Q = \int \rho ds^2 = \sum_{l=1}^N W_l = \sum_{l=1}^N \beta_l \eta_l, \quad (12)$$

where  $W_l$  is just the winding number of  $\vec{Z}$  around  $z_l$ , the above expression reveals distinctly that the topological charge of spiral wave is not only the winding number, but also expressed by the Hopf indices and Brouwer degrees. For multi-armed spiral wave, the Hopf indices  $\beta_l$  characterizes the numbers of arms in a spiral wave ( $\beta_l = 1$ , a one-armed spiral wave;  $\beta_l = 2$ , a two-armed spiral wave). And the Brouwer degree  $\eta_l$ , whose sign is very important, characterizes the direction of rotation of spiral wave ( $\eta_l = +1$ , the spiral wave rotating clockwise;  $\eta_l = -1$ , the spiral wave rotating anti-clockwise). It is seen that our topological theory provides a more detailed description of the inner structures of spiral wave. The topological inner structure showed in Eq. (11) is more essential than that in Eq. (2), this is just the advantage of our topological description of the spiral wave.

According to Eq.(3), it is easy to see that the topological current  $J^i$  is identically conserved,

$$\partial_i J^i = 0. \quad (13)$$

This equation implies the conservation of the topological charge of spiral wave:

$$\partial_t \rho + \nabla \cdot \vec{J} = 0, \quad (14)$$

which is only the topological property of the complex function  $Z$ . The conservation of the total topological charge  $Q$  is a very stronger topological constraint on spiral wave, many important properties of spiral waves due to this topological rule. In our following sections, we discuss the generating, annihilating at the limit points and encountering, splitting, merging at the bifurcation points of complex function  $Z$ , and it shows that the conservation of the total topological charge  $Q$  is also valid in these processes.

### III. THE GENERATION AND ANNIHILATION OF SPIRAL WAVES

As investigated before, the zeros of the complex function  $Z$  play an important role in describing the topological structures of spiral waves. Now we begin discussing the properties of the zero points, in other words, the properties of the solutions of Eq. (7). As we knew before, if the Jacobian

$$D^0\left(\frac{\phi}{x}\right) \neq 0, \quad (15)$$

we will have the isolated zeros of the vector field  $\vec{Z}$ . The isolated solutions are called regular points. However, when the condition (15) fails, the usual implicit function theorem [32] is of no use. The above discussion will change in some way and will lead to the branch process. We denote one of the zero points as  $(t^*, \vec{z}_l)$ . If the Jacobian

$$D^1\left(\frac{\phi}{x}\right) \Big|_{(t^*, \vec{z}_l)} \neq 0, \quad (16)$$

we can use the Jacobian  $D^1(\frac{\phi}{x})$  instead of  $D^0(\frac{\phi}{x})$  for the purpose of using the implicit function theorem. Then we have a unique solution of Eq.(7) in the neighborhood of the limit point  $(t^*, \vec{z}_l)$

$$t = t(x^1), \quad x^2 = x^2(x^1), \quad (17)$$

with  $t^* = t(z_l^1)$ . We call the critical points  $(t^*, \vec{z}_l)$  the limit points. In the present case, we know that

$$\frac{dx^1}{dt} \Big|_{(t^*, \vec{z}_l)} = \frac{D^1(\frac{\phi}{x})}{D(\frac{\phi}{x})} \Big|_{(t^*, \vec{z}_l)} = \infty \quad (18)$$

i.e.,

$$\frac{dt}{dx^1} \Big|_{(t^*, \vec{z}_l)} = 0 \quad (19)$$

Then the Taylor expansion of  $t = t(x^1)$  at the limit point  $(t^*, \vec{z}_l)$  is

$$t - t^* = \frac{1}{2} \frac{d^2 t}{(dx^1)^2} \Big|_{(t^*, \vec{z}_l)} (x^1 - z_l^1)^2, \quad (20)$$

which is a parabola in  $x^1 - t$  plane. From Eq.(20) we can obtain two solutions  $x_1^1(t)$  and  $x_2^1(t)$ , which give two branch solutions (world lines of the spiral waves). If

$$\frac{d^2 t}{(dx^1)^2} \Big|_{(t^*, \vec{z}_l)} > 0, \quad (21)$$

We have the branch solutions for  $t > t^*$  [see Fig.1(a)]; otherwise, we have the branch solutions for  $t < t^*$  [see Fig.1(b)]. These two cases are related to the origin and annihilation of the spiral waves.

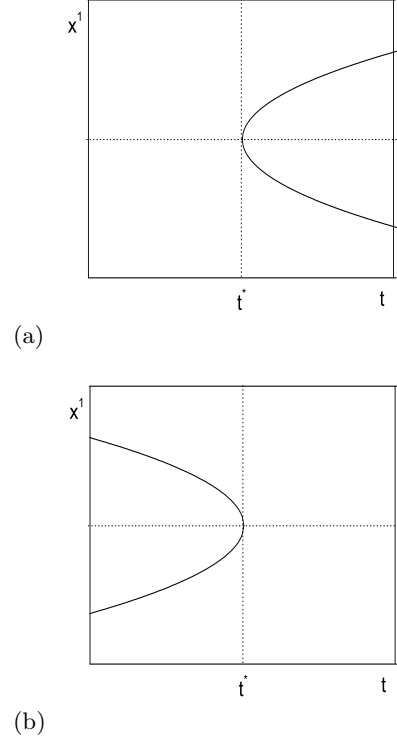


FIG. 1: Projecting the world lines of spiral waves onto  $(x^1 - t)$  plane. (a) The branch solutions for Eq.(20) when  $d^2 t / (dx^1)^2|_{(t^*, \vec{z}_l)} > 0$ , i.e., a pair of spiral waves with opposite charges generate at the limit point, i.e., the origin of spiral waves. (b) The branch solutions for Eq.(20) when  $d^2 t / (dx^1)^2|_{(t^*, \vec{z}_l)} < 0$ , i.e., a pair of spiral waves with opposite charges annihilate at the limit point.

One of the results of Eq.(18), that the velocity of the spiral waves are infinite when they are annihilating, agrees with the fact obtained by Bray [34] who has a scaling argument associated with the point defects final annihilation which leaves a large velocity tail. From Eq.(18), we also obtain a new result that the velocity field of spiral waves is infinite when they are generating, which is gained only from the topology of the complex function  $Z$ .

Since topological current is identically conserved, the topological charge of these two generated or annihilated spiral pair must be opposite at the limit point, i.e.,

$$\beta_{l_1} \eta_{l_1} = -\beta_{l_2} \eta_{l_2} \quad (22)$$

which shows that  $\beta_{l_1} = \beta_{l_2}$  and  $\eta_{l_1} = -\eta_{l_2}$ . One can see that the fact the Brouwer degree  $\eta$  is indefinite at the limit points implies that it can change discontinuously at limit points along the world lines of the spiral vortices (from  $\pm 1$  to  $\mp 1$ ). It is easy to see from Fig. 1: when  $x^1 > z_l^1$ ,  $\eta_{l_1} = \pm 1$ ; when  $x^1 < z_l^1$ ,  $\eta_{l_1} = \mp 1$ .

For a limit point it is required that  $D^1(\frac{\phi}{x})|_{(t^*, z_l^1)} \neq 0$ . As to a bifurcation point [35], it must satisfy a more complex condition. This case will be discussed in the following section.

#### IV. BIFURCATION OF THE VELOCITY FIELD OF THE PHASE SINGULARITY FOR SPIRAL WAVE

In this section we have the restrictions of Eq. (7) at the bifurcation points  $(t^*, z_l^1)$ ,

$$D(\frac{\phi}{x})|_{z_l^1} = 0, \quad D^1(\frac{\phi}{x})|_{z_l^1} = 0, \quad (23)$$

which leads to an important fact that the function relationship between  $t$  and  $x^1$  is not unique in the neighborhood of the bifurcation point  $(t^*, z_l^1)$ . It is easy to see that

$$V^1 = \frac{dx^1}{dt} = \frac{D^1(\frac{\phi}{x})}{D(\frac{\phi}{x})}|_{z_l^1}, \quad (24)$$

which under Eq.(23) directly shows that the direction of the integral curve of Eq.(24) is indefinite at  $(t^*, z_l^1)$ , i.e., the velocity field of the phase singularity for spirals is indefinite at  $(t^*, z_l^1)$ . That is why the very point  $(t^*, z_l^1)$  is called a bifurcation point.

Assume that the bifurcation point  $(t^*, z_l^1)$  has been found from Eqs.(7) and (23). We know that, at the bifurcation point  $(t^*, z_l^1)$ , the rank of the Jacobian matrix  $[\frac{\partial \phi}{\partial x}]$  is 1. In addition, according to the  $\phi$ -mapping theory, the Taylor expansion of the solution of  $\phi^1$  and  $\phi^2$  in the neighborhood of the bifurcation point can generally be denoted as

$$A(x^1 - z_l^1)^2 + 2B(x^2 - z_l^2)(t - t^*) + (t - t^*)^2 = 0, \quad (25)$$

which leads to

$$A(\frac{dx^1}{dt})^2 + 2B\frac{dx^1}{dt} + C = 0 \quad (26)$$

and

$$C(\frac{dt}{dx^1})^2 + 2B\frac{dt}{dx^1} + A = 0, \quad (27)$$

where  $A$ ,  $B$ , and  $C$  are three constants. The solution of Eq.(26) or Eq.(27) give different directions of the branch curves (world lines of the spiral waves) at the bifurcation point. There are four kinds of important cases, which will show the physical meanings of the bifurcation points.

Case 1( $A \neq 0$ ). For  $\Delta = 4(B^2 - AC) > 0$ , we get two different directions of the velocity field of the phase singularity for spiral waves

$$\frac{dx^1}{dt}\bigg|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A} \quad (28)$$

which is shown in Fig.2. It is the intersection of two phase singularity for spirals, which means that two phase singularity for spirals meet and then depart from each other at the bifurcation point.

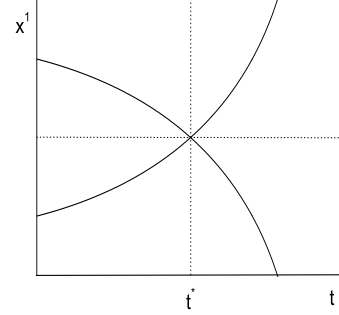


FIG. 2: Projecting the world lines of spiral waves onto  $(x^1 - t)$  plane. Two spiral waves meet and then depart at the bifurcation point.

Case 2( $A \neq 0$ ). For  $\Delta = 4(B^2 - AC) = 0$ , the direction of the velocity field of the phase singularity is only one

$$\frac{dx^1}{dt}\bigg|_{1,2} = \frac{-B}{A} \quad (29)$$

which includes three important situations. (a) One world line resolves into two world lines, i.e., one spiral wave splits into a spiral pair at the bifurcation point [see Fig.3(a)]. (b) Two world lines merge into one world line, i.e., a spiral pair merge into one spiral at the bifurcation point [see Fig.3(b)]. (c) Two world lines tangentially contact, i.e., a spiral pair tangentially encounter at the bifurcation point [see Fig.3(c)].

Case 3( $A = 0, C \neq 0$ ). For  $\Delta = 4(B^2 - AC) = 0$ , we have

$$\frac{dt}{dx^1}\bigg|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0, \quad -\frac{2B}{C} \quad (30)$$

There are two important cases: (a) Three world lines merge into one world line, i.e., three spirals merge into a spiral at the bifurcation point [see Fig.4(a)]. (b) One world line resolves into three world lines, i.e., a spiral splits into three spirals at the bifurcation point [see Fig.4(b)].

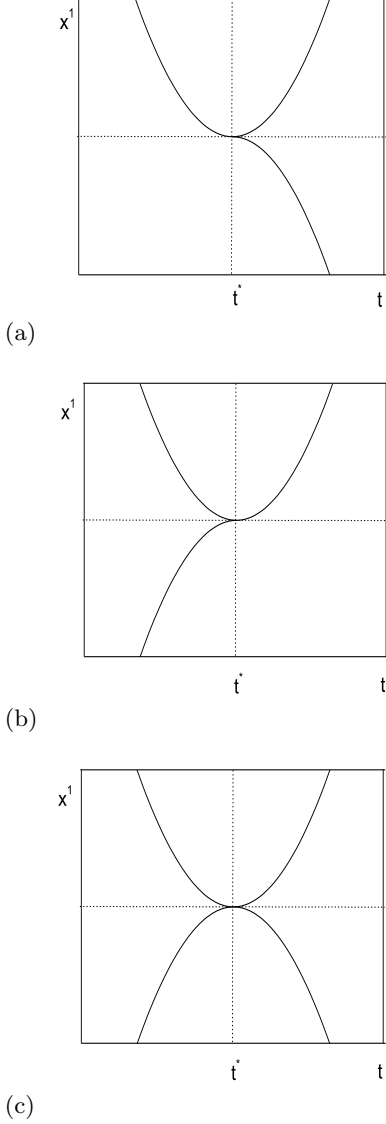


FIG. 3: (a) One spiral wave splits into two spiral wave at the bifurcation point. (b) Two spiral waves merge into one spiral wave at the bifurcation point. (c) Two world line of spiral waves tangentially intersect, i.e., two spiral waves tangentially encounter at the bifurcation point.

Case 4 ( $A = C = 0$ ). Equation(26) and Eq(27) give respectively

$$\frac{dx^1}{dt} = 0, \quad \frac{dt}{dx^1} = 0. \quad (31)$$

This case is obvious, see Fig.5, and is similar to Case 3.

The above solution reveal the evolution of the spirals. Besides the encountering of the spiral waves, i.e., a spiral pair encounter and then depart at the bifurcation point along different branch cures [see Fig.2 and Fig.3(c)], it also includes splitting and merging of spirals. When a multi-charged spiral moves through the bifurcation point,

it may split into several spirals along different branch curves [see Fig.3(a), Fig.4(b), Fig.5(b)]. On the contrary, several spirals can merge into a spiral at the bifurcation point [see Fig.3(b) and Fig.4(a)].

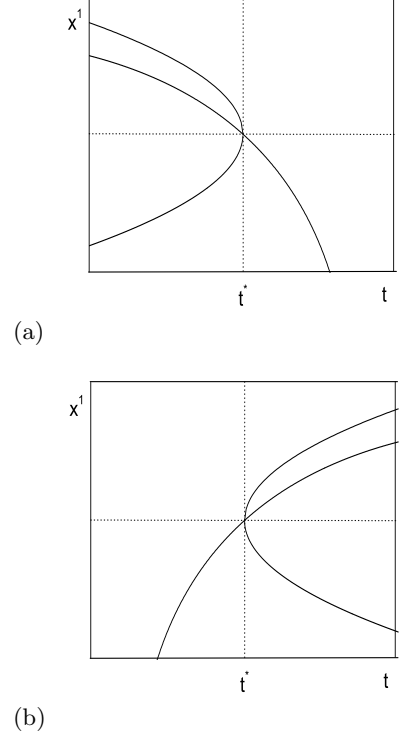


FIG. 4: Two important cases of Eq.(30). (a) Three spiral waves merge into one at the bifurcation point. (b) One spiral wave splits into three spiral waves at the bifurcation point.

The identical conversation of the topological charge shows the sum of the topological charge of these final spirals must be equal to that of the original spirals at the bifurcation point, i.e.,

$$\sum_i \beta_{l_i} \eta_i = \sum_f \beta_{l_f} \eta_f \quad (32)$$

for fixed  $l$ . Furthermore, from the above studies, we see that the generation, annihilation, and bifurcation of spirals are not gradually changed, but suddenly changed at the critical points.

## V. APPLICATION AND DISCUSSION

Our conclusions can be summarized as follows: First, by making use of *Duan's topological current theory*, we obtain the charge density of spiral waves and the topological inner structure of its topological charge, the spiral waves can be classified not only by their topological charges, but especially importantly by their Hopf indices

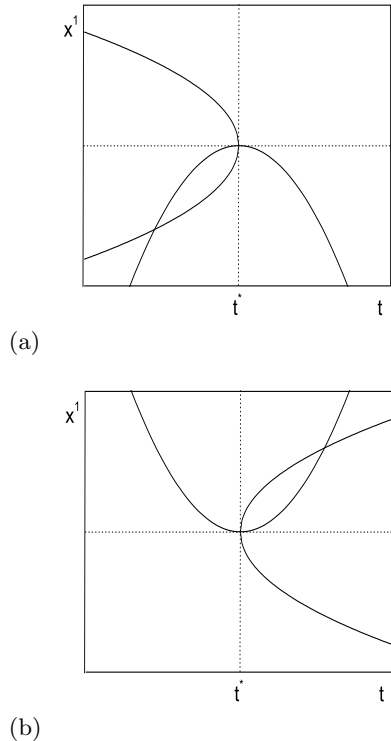


FIG. 5: Two world lines intersect normally at the bifurcation point. This case is similar to FIG. 4. (a) Three spiral waves merge into one at the bifurcation point. (b) One spiral wave splits into three spiral waves at the bifurcation point.

and Brouwer degrees. This is more general than usually considered and will be helpful as a complement of the current description of spiral wave only by the winding numbers in topology. Second, the evolution of spiral wave is also studied from the topological properties of a two-dimensional vector field  $\vec{Z}$ . We find that there exist crucial cases of branch processes in the evolution of the spiral wave when  $D(\frac{\phi}{x}) \neq 0$ , i.e.,  $\eta_l$  is indefinite. This means that the spiral waves are generate or annihilate at the limit points and encounter, split, or merge at the bifurcation points of the two-dimensional vector field  $\vec{Z}$ , which shows that the spiral wave system is unstable at these branch points. Third, we found the result that

the velocity of spiral core is infinite when they are annihilating or generating, which is obtained only from the topological properties of the two-dimensional vector field  $\vec{Z}$ . Forth, we must pointed out that there exist two restrictions of the evolution of spiral waves. One restriction is the conservation of the topological charge of the spiral waves during the branch process [see Eqs. (22) and (32)], the other restriction is that the number of different directions of the world lines of spiral waves is at most 4 at the bifurcation points [see Eqs. (26) and (27)]. The first restriction is already known, but the second is pointed out here for the first time to our knowledge. We hope that it can be verified in the future. Finally, we would like to point out that all the results in this paper have been obtained only from the viewpoint of topology without using any particular models or hypothesis and do not require the knowledge of the actual solution of the reaction-diffusion equation.

In this paper, we give a rigorous and general topological investigation of spiral wave. This work can be applied in several ways. First, topology and geometry of two-dimensional manifold in which the spiral lived can affect the dynamics of spiral wave[8, 9], therefore, it is very interesting to extend our work to a curved surface which have complex topology and geometry, and study the interaction between the topology of spiral wave and the topology of curved surface. Second, our theory can be easily extend to the three dimensional scroll wave, which is just a three dimensional analog of spiral wave, in this situation, a more complex topology related to knots must be considered[29]. Third, our work provide a more detailed description of spiral wave, the detailed information of spiral can applied to the study of dynamics of spiral wave. These issues are very worthwhile to consider and will be investigated in our further works.

### Acknowledgments

Thanks to the works of Dr. X. H. Zhang and Dr. B. H. Gong in drawing the figures in this paper. This work was supported by the National Natural Science Foundation of China and the Cuiying Programme of Lanzhou University.

- 
- [1] A. N. Zaikin and A. M. Zhabotinsky, *Nature* **225**, 535 (1970).
  - [2] A.T. Winfree, *Science* **175**, 634 (1972).
  - [3] Th. Plessner, S. C. Müller and B. Hess, *J. Phys. Chem.* **94**, 7501 (1990).
  - [4] J. M. Davidenko, A. V. Pertsov, R. Salomonsz, W. Baxter and J. Jalife, *Nature* **355**, 349 (1992).
  - [5] S. Jakubith, H. H. Rotermund, W. Engel, A. von Oertzen and G. Ertl, *Phys. Rev. Lett.* **65**, 3013 (1990).
  - [6] K. I. Agladze and V. I. Krinsky, *Nature* **296**, 424 (1982);

- V. I. Krinsky and K. I. Agladze, *Dokl. Akad. Nauk SSSR* **263**, 335 (1982).
- [7] B. Vasiev, F. Siegert, and C. Weijer, *Phys. Rev. Lett.* **78**, 2489 (1997).
- [8] J. Davidsen, L. Glass, and R. Kapral, *Phys. Rev. E* **70**, 056203 (2004).
- [9] K. Rohlf, L. Glass, and R. Kapral, *Chaos* **16**, 037115 (2006).
- [10] H. Zhang, Z. Cao, N. J. Wu, H. P. Ying, and G. Hu, *Phys. Rev. Lett.* **94**, 188301 (2005).

- [11] H. Zhang, B. Hu, G. Hu, and J. Xiao, J. Chem. Phys. **121**, 7276 (2004); J. X. Cheng, H. Zhang, and Y. Q. Li, J. Chem. Phys. **124**, 014505 (2005).
- [12] H. Zhang, B. Hu, B. W. Li, and Y. S. Duan, Chin. Phys. Lett. **24**, 1618 (2007).
- [13] A. T. Winfree, *The Geometry of Biological Time* (Springer-Verlag, New York, 2001).
- [14] L. Glass, Science **198**, 321 (1977).
- [15] A. T. Winfree and S. H. Strogatz, Physica D **8**, 35 (1983); Physica D **9**, 65 (1983); Physica D **9**, 335 (1983); Physica D **13**, 221 (1984).
- [16] F. Liu and F. G. Mazenko, Phys. Rev. B **46**, 5963 (1992).
- [17] D. Sumners, in *Graph Theory and Topology in Chemistry*, edited by R. B. King and D. Rouvray (Elsevier, Amsterdam, 1987), p. 3.
- [18] I. Cruz-White, Ph. D. thesis, Florida State University, 2003.
- [19] A. T. Winfree, Sci. Am. **230**, 82 (1974).
- [20] P. J. Nandapurkar and A. T. Winfree, Physica D **29**, 69 (1987).
- [21] A. T. Winfree, *When Time Breaks Down* (Princeton University Press, 1987).
- [22] A. T. Winfree, SIAM Rev. **32**, 1 (1990).
- [23] R. A. Gray, A. M. Pertsov, and J. Jalife, Nature **392**, 75 (1998).
- [24] A. T. Winfree, Nature **371**, 233 (1994); Physica D **84**, 126 (1995).
- [25] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, Heidelberg, 1984).
- [26] A. S. Mikhailov, *Foundations of Synergetics I: Distributed Active Systems* (Springer-Verlag, New York, 1994).
- [27] L. B. Fu, Y. S. Duan, and H. Zhang, Phys. Rev. D **52**, 045004 (2000).
- [28] Y. S. Duan, S. L. Zhang, and S. S. Feng, J. Math. Phys. **35**, 4463 (1994).
- [29] J. R. Ren, T. Zhu, and Y. S. Duan, Chin. Phys. Lett. **25**, 353 (2008) [arXiv:0712.4196].
- [30] S. F. Mo, J. R. Ren, and T. Zhu, J. Phys. A: Math. Theor. **41**, 315214 (2008) [arXiv:0807.2784]; W. K. Qi, T. Zhu, Y. Chen, and J. R. Ren, arXiv:0805.4661; J. R. Ren, T. Zhu, and S. F. Mo, arXiv:0712.4198; J. R. Ren, T. Zhu, and Y. S. Duan, Commun. Theor. Phys. **50** (2008)345-348 [arXiv:0712.4195]; J. R. Ren, T. Zhu, and Y. S. Duan, Chin. Phys. Lett. **25**, 367-370 (2008) [arXiv:0707.4529].
- [31] Y. S. Duan, X. Liu, and L. B. Fu, Phys. Rev. D **67**, 085022 (2003).
- [32] E. Goursat, *A Course in Mathematical Analysis*, translated by E. R. Hedrick (Dover, New York, 1904), Vol. I.
- [33] J. A. Schouton, *Tensor Analysis for Physicists* (Clarendon, Oxford, 1951).
- [34] A. J. Bray, Phys. Rev. E **55**, 5297 (1997).
- [35] M. Kubicek and M. Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures* (Springer-Verlag, New York, 1983).